Entanglements in glass*

Nicolas Rivier
Blackett Laboratory, Imperial College, GB London SW7 2BZ, UK
and Institut de Physique Expérimentale, Université de Lausanne,
1015 Dorigny-Lausanne, Switzerland

Ground state and elementary excitations (tunnelling modes) in glass are obtained from an analysis of its symmetry, a local gauge invariance. The configuration of glass is represented as a discrete fiber bundle. The base space is a continuous random network, standard model of the structure of covalent glasses. The connection is determined naturally by the elasticity of the network. The bundle is non-trivial, the elastic connection is entangled in one of two ways. Sources of non-triviality are closed loops, threading through odd rings in the network. To restore gauge invariance, tunnelling must occur between the two possible configurations about an odd loop. Entanglement and elementary excitations are labelled by permutations of the covalent bonds incident on an atom.

1. Introduction

It is not the structure of glass which is entangled, but its physical properties. This entanglement manifests itself through extraordinary behaviour (decoupled additional excitations: tunnelling modes [1]), which is a signature (specific and universal) of disordered condensed matter at low temperature. The mathematical mechanism by which a harmless (albeit random) structure induces frustrated[1)] [2] behaviour in the physical quantities attached to it is called a *fiber bundle*. Indeed, glass can be regarded as the archetype of a non-trivial fiber bundle in condensed matter, where the base space is the structure of the material itself [3]. Structural defects, the source of non-triviality of the bundle, are then directly responsible for the extraordinary physics.

A disordered structure is homogeneous and isotropic, but not in a generative fashion: after some displacement, the local environment is different, but this difference (fluctuation) is not objective, it just happens here. Glass, as a whole, is a homogeneous, space-filling collection of elements (Vierbein) fluctuating in orientation. Automorphisms probing this overall homogeneity are not displacements, but local transformations

* Work supported by the Herbette Foundation.
1) The label "frustration" is due to P.W. Anderson.

(rotation and reflection of the Vierbein) which leave unchanged the global structural and physical properties of the material. These automorphisms are gauge transformations, and the overall symmetry of glass is *gauge invariance* [3]. Now, the natural geometrical framework for gauge symmetry is a fiber bundle. (Fiber bundle is defined briefly in section 3 as the geometrical structure of a physical system; it is not an abstract mathematical construction.)

2. Architecture and symmetry of glass

Glass (for example, vitreous silica SiO_2 (window glass)) is represented by a solid random network. Its vertices $\{V\}$ are the Si atoms. Chemistry (or quantum mechanics) requires that they are tetravalent: every vertex has valency or coordinance $z = 4$ and the network is a regular graph. Short-range order is perfect (ignoring "dangling bonds") and the regular graph is called a continuous random network.

Chemical bonds Si–O–Si, decorated by oxygen, are the edges $\{E\}$ of the network. Bonds can be bent or stretched, at an energy cost, so that the vertices remain perfect tetrapods or *Vierbein*, while the overall network is random. (Vierbein is both singular and plural, with apologies to German grammar.) Randomness sets in very rapidly from the perfect short-range order of the Vierbein; indeed, there is no long-range order from 4 nm (about 10 interatomic Si–Si distances) out. This distance turns out to be also the screening length for elastic stresses in the glass.

In spite of being based on a discrete network, glass is therefore extremely disordered and homogeneous: "... One of the most interesting discoveries made in the comparatively early history of X-ray analysis was the fact that silk or even paper are more crystalline than glass." (Kathleen Lonsdale). This homogeneity (a non-generative symmetry in a random medium: lost in a forest, you may walk from tree to tree and face a different species, but this information does not help you to find your way out) is a gauge invariance [3]. The local reference frame (Vierbein in glass) has arbitrariness, which is expressed as a gauge invariance (invariance of the physical properties of the glass with respect to a local or gauge transformation).

Faces of the network are (shortest) rings, usually puckered, but not entangled. We shall see that these planar rings are sources of physical entanglement, which is measurable. Cells have no direct geometrical interpretation.

Glass is a form of condensed matter. It has a spacial structure and is in a fixed configuration. This implies that the local reference frame (Vierbein) must be returned to the same orientation or to an equivalent one after circumnavigation. Circumnavigation is determined by parallel transport, but here on a discrete network (physically imposed connection on a discrete fiber bundle [4]). This restriction is a geometrical quantization, also seen in dislocation in crystals where the quantized equivalent configurations are labelled by the (space) group of symmetry of the crystalline structure. In glass, as in any disordered structure, the space group is trivial (it consists of the identity operation only), but we shall see that there are non-trivial configurations in which the transported Vierbein is entangled, and these non-trivial configurations make up the ground state of the material. (By contrast, the "geometric (Berry) phase " is not quantized because the system is not part of a condensed structure. For example, the rotation angle of a Foucault pendulum is a continuous function of the latitude.)

Entanglements in glass are physical and measurable. They are caused by the presence of non-trivial rings. The ring itself is planar, but circumnavigation of the Vierbein around it is not flat. Non-trivial rings (rings with an odd number of bonds) are sources of non-triviality of the fiber bundle representing the (gauge-invariant) glass: the base space is the network itself, and the connection is imposed by physics.

Section 3 presents glass as a fiber bundle. It is not essential in order to understand the rest of the paper. The main points are [3]:

(i) Gauge invariance is the symmetry of disorder, and the natural geometric framework of gauge theories is a fiber bundle.

(ii) In condensed matter, the base space of the bundle is the geometrical network of atoms and bonds. In glass, this network is a regular, random graph (continuous random network).

(iii) The sources of non-triviality (loci of frustration) of the bundle are odd rings. Odd rings do not occur in isolation, but are threaded through by uninterrupted lines which form closed loops or terminate at the surface of the material [5]. Thus, glass is constituted of necklaces of odd rings, similar to vortices or dislocations, but characterized by existence (oddness) instead of intensity. (This result is a geometric conservation law (Poincaré's identity) expressing the fact that a boundary (∂) has no boundary ($\partial\partial \equiv 0$). An odd ring is a boundary (since a ring with zero (even) bonds is manifestly bounding, and the number of bonds is not a conserved quantity in the absence of generative rotation symmetry – it is changed by a disclination). Proof of this conservation law is elementary [5,3].)

3. Glass as a fiber bundle

Gauge invariance is the symmetry of disorder. Glass is a disordered solid and can be represented as a fiber bundle because a gauge field, geometrically, is a connection in a fiber bundle. A fiber bundle consists of a *base space* (the regular graph of Si atoms and bonds), a total space (position and relative orientation of the Vierbein), and a map that projects every point of the total space onto a point in the base space. The set of all points in the total space that are mapped onto the same point in the base is called the *fiber.*

(In most condensed matter applications, the base space is the ordinary space occupied by the material, and the fiber represents "the parameter to be gauged". In glass, the fiber consists of the set of permutations of the legs of a Vierbein. It will be defined naturally below.)

The *connection* or *gauge field* describes how the orientation of the Vierbein changes as one goes along a path in base space, or how the path in base space is *lifted* into a path of the fiber bundle. In glass, this connection is a natural (physical) one: it is carried by the (bond-bending) energy between two Vierbein separated by a bond. (See Bernstein and Phillips [6] for other examples, and for reassurances as to the naturalness of the connection process.)

When the base space is continuous, the connection is usually carried through parallel transport, a prescription that the orientation of the fiber (or of any transported object) remains constant along a geodesic in base space. This is the basis of a macroscopic description of glass as an elastic continuum (disordered) [7].

A trivial fiber bundle is one for which the total space is the direct product of the base space by the fiber. Conversely, a bundle is not trivial if it ends up twisted or entangled: the position on the fiber is modified as one moves along a close contour (ring) in base space. The fiber at the end of the circumnavigation is indistinguishable from that at the beginning (because there is only one, unique representative fiber, but also, physically, because we are in a solid with a given, frozen configuration). Yet, the successive connections end up entangled. Such a twist or entanglement is called a *large gauge transformation.* It is topologically stable and cannot be undone by infinitesimal corrections along the contour. Glass is entangled around odd rings, as we shall see.

The *curvature* of the region in the base surrounded by a closed path is given by the distance along any one of the fibers made by the lifted path. In glasses, fiber bundles on discrete networks, the natural path lifting operation generalizes the notion of curvature without requiring parallel transport. Thus, an even ring is flat and an odd ring is curved [5].

Glasses have non-generative symmetry, probed by automorphisms which act as translations along the fiber. The problem is to separate the degree of freedom into those labelling the base and those labelling the fiber, and to define the projection. Bundles describing topologically disordered materials are highly non-trivial (there is a lot of frustration in glasses (a large density of odd lines)), so that the separation between "internal" (fiber) and base parameters is not obvious, and this non-triviality leads to remarkable physical properties, which we now encounter.

4. Physical properties of glass, demonstrating entanglement

Glasses have anomalous physical properties – specific heat, phonon transport, saturability and echoes (coherence) – below 1 K [1], which can be modelled by tunnelling modes between pairs of potential valleys in configuration space, represented schematically (but accurately as far as the energy scale and the physical properties are concerned) in fig. 1. The specific heat is approximately linear in temperature below 1 K, where a linear function exceeds the T^3 contribution of phonons in a three-dimensional system. There are therefore additional elementary excitations in glass besides phonons. A distribution of two-level systems of fig. 1 yields a linear specific heat (as for electrons in metals: Boltzmann's distribution on a two-level system is Fermi–Dirac's, and the distribution $P(\Delta_0)$ of energy spacings plays the part of the density of electronic states). Heat is carried by phonons. Glass has a lower

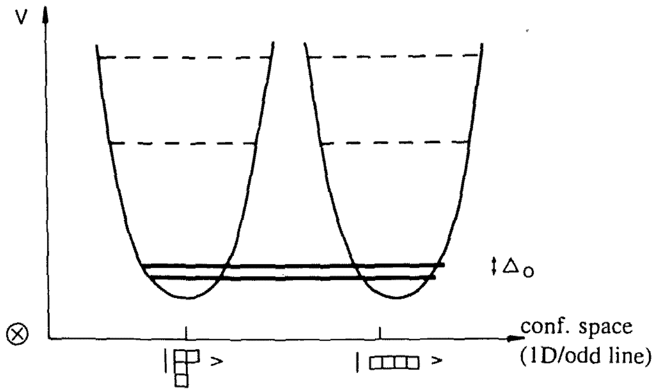


Fig. 1. Potential valleys in configuration space, giving rise to tunnelling modes. The configuration space is a direct product of one-dimensional subspaces, each associated with one odd line. Topology of the valleys is sufficient to obtain (qualitatively) ground state and elementary excitations in glass [3].

thermal conductivity than its corresponding crystalline material (e.g. quartz), so the phonons are absorbed by the additional excitations, as they should be in the model of fig. 1. If the power of the transducer is increased, most two-level systems will have absorbed a phonon and be in their excited state. They can no longer absorb phonons, and the thermal conductivity of the crystal is recovered. Finally, an echo can be set up on these excitations, as in any quantum two-level system (spin 1/2). The echo delay time is approximately 10 μs at 20 mK, which shows that the two-level systems are effectively decoupled (the dephasing time T_2 is greater than this very long delay time). Saturation and echo also indicate that the two-level excitations are due to quantum tunnelling (classical, thermal excitations are at much higher energy, out of reach at these low temperatures). In fig. 1, the energy splitting Δ_0 of the two levels is due entirely to tunnelling between degenerate potential wells. Any departure from degeneracy Δ yields an additional energy difference ($>\Delta$ if $\Delta > \Delta_0$).

Apart from the fact that one does not know precisely what tunnels, the presence of nearly degenerate (to within less than 10^{-4} eV) classical ground states (potential minima) in a system with no obvious symmetry to impose the degeneracy is astonishing: bulk condensed matter usually has one single ground state, and its potential energy, one single minimum in a many-dimensional configuration space. Excitations about this minimum are phonons, which can also be heard in glass. (By contrast, multiply-connected condensed matter can accommodate several, nearly degenerate, low-energy configurations. For example, the energy of a superconducting ring trapping n magnetic flux quanta depends only on n through the contribution of the magnetic field energy outside the ring. The energy inside the material is identical for all n.)

Observation of echo (analogous to spin echo, but generated by sound pulses) suggests that different tunnelling modes – different pairs of potential valleys in fig. 1 – are uncoupled enough to preserve phase coherence between excitation and echo ($\approx 10\,\mu$s at 20 mK). What are these tunnelling modes, why are they decoupled, and why are they degenerate in a strongly correlated system of many atoms with trivial space group?

The answers can be found in the elasticity of continuous random networks. The normal modes are phonons and the additional excitations of fig. 1. The lowest elastic energy spectrum of continuous random networks about odd lines is that of fig. 1 [8]. Even rings have standard spectrum, with one single potential minimum. By contrast, each odd line has (in a $z = 4$ network) two alternative ground-state configurations, degenerate in energy. The degeneracy is due to gauge invariance.

5. Elasticity of random networks

Consider a solid, continuous random network, made of perfect Vierbein $\{V\}$ and stretchable and bendable bonds $\{E\}$. The elastic energy (eq. (1) below) is carried by the bonds connecting two Vierbein. It establishes the connection, as described below.

There are two types of potential energies, bond-stretching and bond-bending. Bond-stretching is very much the stronger of the two, but neglecting bond-bending altogether leaves the network underconstrained (wobbly), with as many zero frequency modes as there are atoms if the network is tetracoordinated. These zero frequency modes are degenerate ground states. Let us see how and whether this degeneracy is lifted by bond-bending forces.

5.1. BOND-STRETCHING ONLY

The elastic potential energy of N atoms in a network consists of two terms [9,8]: a strong, bond-stretching V_1 and a weaker, bond-bending contribution V_2. V_1 can be written in terms of the relative displacement of two neighbouring atoms only. V_2 requires the direction towards a third atom as well. It is still a two-body potential, but the "bodies" are tetrapods,

$$V = V_1 + V_2 = (W_1/2) \sum_{i\alpha} [(r_i - r_{i\alpha}) \cdot u_{i\alpha}]^2 + (W_2/4) \sum_{i(\alpha\beta)} [(r_i - r_{i\alpha}) \cdot u_{i\beta} + (r_i - r_{i\beta}) \cdot u_{i\alpha}]^2. \tag{1}$$

Here, $\boldsymbol{r}_i$ is the displacement of atom i, $\boldsymbol{r}_{i\alpha}$ that of its neighbour along direction α, and $\boldsymbol{u}_{i\alpha}$ is the direction of bond $i\alpha$ before displacement ($\alpha = 1, \ldots, 4$). $i(\alpha\beta)$ labels two bonds $i\alpha$ and $i\beta$ incident on the same vertex i.

Neglecting V_2 in a first stage, we find ([9]; [8] has the simplest derivation (following Mosseri)) normal modes split into three groups, a band of N phonons, flanked (for tetravalent networks) by two sets of N degenerate modes each (fig. 2). Since we are only interested in the ground-state configuration(s) and in the lowest energy excitations of the material, we concentrate on the N floppy modes at $\omega = 0$.

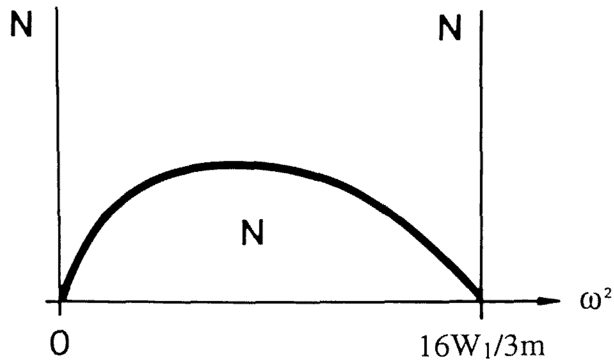


Fig. 2. Schematic spectrum of the normal, bond-stretching modes of a $z = 4$ covalent network representing a-Si [9,3]. The N floppy ($\omega = 0$) modes are clearly distinguished from the band of (N) phonons.

Accounting for these N floppy modes is the same as for independent currents in an electrical network [10]: they lie on edges not on a spanning tree of the network, and each independent edge closes an independent circuit or ring. Bond $i\alpha$ remains unstretched if $q_{i\alpha} = \boldsymbol{r}_i \cdot \boldsymbol{u}_{i\alpha} = \boldsymbol{r}_{i\alpha} \cdot \boldsymbol{u}_{i\alpha}$. Hence, an $\omega = 0$ mode is characterized by a scalar edge (bond) variable $q_{i\alpha}$, as is a current through a link of an electrical network. Moreover, equilibrium requires $\sum_\alpha \boldsymbol{u}_{i\alpha} = 0$, thus $\sum_\alpha q_{i\alpha} = 0$, which is Kirchhoff's current law. There is also a ring closure relation analogous to Kirchhoff's voltage law, but it is unnecessary to count independent modes [10,8]. With $E = (z/2)N$ edge variables $q_{i\alpha}$ and N current law constraints, the number of $\omega = 0$ modes is N for $z = 4$. (There are $R_1 = E - N + 1$ independent circuits, where R_1 is the cyclomatic, or first Betti number, a topological invariant of the graph. The additional mode is the $\omega = 0$ of the phonon band (rigid translation).)

Note that simplistic mode counting – $3N$ elastic modes, with $E = 2N$ constraints, $\omega = 0$ on edges – gives accidentally the correct number. It is correct only if the constraints are independent, which they are not. In any case, Kirchhoff does construct the independent modes, besides counting them. Note also that the choice of spanning tree is arbitrary.

5.2. INCLUDING BOND-BENDING FORCES. CONNECTION

The problem is to find which independent rings (listed by Kirchhoff's spanning tree construction) are trivial (have a unique ground state) and which are not (retain several stable configurations) when bond-bending forces are present. Note that an odd loop in the network (line threading odd rings) can be represented by an "air tube", a torus subgraph with cross sections made up of the odd rings threaded through by the odd line and with even rings on its surface. A torus can be laid flat on a plane by making two cuts, one longitudinal, the other transverse, and imposing periodic boundary conditions. The spanning tree can be chosen on the subgraph so that none of its edges crosses these two cuts. All the rings on the plane are even except those broken by the longitudinal cut, which are odd by the definition of an odd line. (The transverse cut may be traversed by odd rings if the odd loop is itself threaded through by an other odd loop, but these odd rings are characteristic of the second loop.) There is only one independent, representative odd ring, which is closed by the bond carrying the independent current.

Let us now include the bond-bending energy V_2, and see how the $N+1$ floppy modes tighten. Specifically, how many of these floppy modes survive as ground states? In order to answer this question, we must describe a configuration and measure its energy. By definition, floppy modes have only bond-bending energy. For a given bond, this energy is measured by comparing the orientations of the two tetrapods which it links or, equivalently, through a congruent transformation of the tetrapod from its orientation at i to that at $i\alpha$. This is a generalization of the bond energy $S_i J_{ij} S_j$ in Ising magnetism, which is given either by comparing the directions of S_i and S_j, or through the congruent transformation (flip or identity) J imposes on the spin; here, the congruent transformation is a rotatory reflection, because bond α is common to the two tetrapods and imposes a mirror reflection. (The rotation part of the transformation is not essential to our argument. It will only affect the spectrum quantitatively.) Recall that we concentrate on floppy modes, described by scalar "currents" $q_{i\alpha}$ on edges $(i\alpha)$ or, more simply, by the congruent transformation or connection. Kirchhoff's formalism remains applicable. The voltage law is imposed by covering transformation of the tetrapod around a circuit. The current law is automatic because the network is connected at labelled Vierbein. (Phonons are distinct modes (the band in fig. 2). Their interaction with tunnelling modes, referred to as phonon localization or as phonon-fraction crossover, is not addressed in this paper.)

Configuration of an n-sided ring is the product of n rotatory reflections. In a given configuration, the tetrapod must be returned to its original orientation (or an equivalent one) after being carried around the ring. The product of rotatory reflections is therefore a covering transformation of the tetrapod, namely a permutation of its legs or of the bond labels. If the ring is even, the permutation, a product of n reflections, is even. If the ring is odd, the permutation is odd. A ring configuration is therefore a path in the discrete fiber bundle returning to the same fiber (it is closed in base space), but not necessarily to the same point on the fiber. The permutation is a gauge transformation moving along the fiber.

It is not the permutation itself which labels the configuration, but only its class: Suppose we were to go around two different rings in succession, recording permutation **R** through one of them, permutation **Q** through the other. The total permutation is $\mathbf{P} = \mathbf{Q} \cdot \mathbf{R}$ or $\mathbf{P}' = \mathbf{R} \cdot \mathbf{Q}$, depending on the order of the circumnavigations and, in general, **P** and **P′** are different. They are related by

$$\mathbf{P}' = \mathbf{R} \cdot \mathbf{P} \cdot \mathbf{R}^{-1} \tag{2}$$

and belong to the same class of the permutation group. The physical configuration, which must be independent of the order of circumnavigations made to measure it, is labelled by the set which includes **P**, **P′**, etc., namely by the *class* of the permutation group to which they belong. The physical fiber bundle identifies in the fiber permutations belonging to the same class (fig. 3).

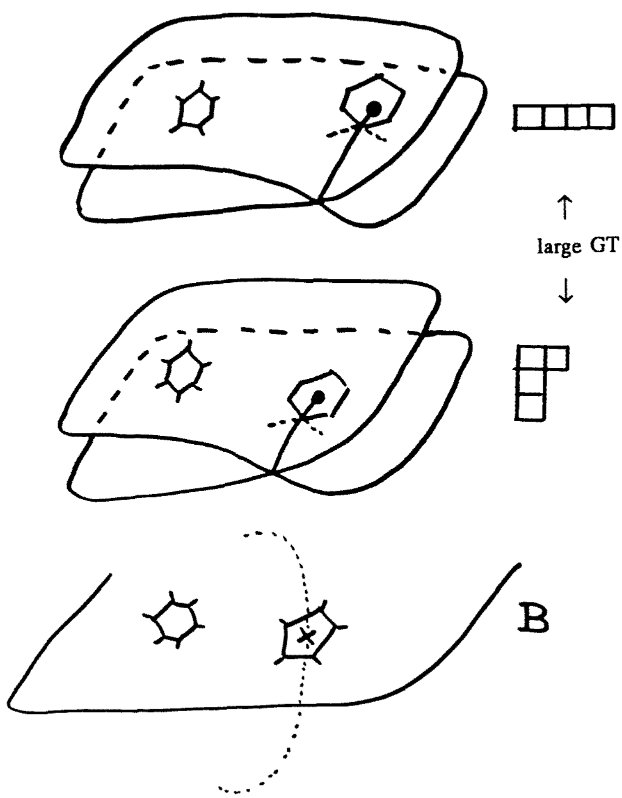
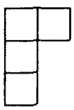
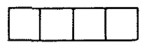


Fig. 3. The fiber bundle for the ground states of an elastic random network with $z = 4$. Here, the total space represents schematically the class of the covering transformation (permutation) of the tetrapod carried around a ring (in base space). The fiber consists of two points. (.....) = odd line, × = odd ring, B = base space.

The ground state of even rings clearly belongs to the identity class of the permutation group of degree $z = 4$, S_4. It is non-degenerate, and even rings are dynamically trivial. Identity permutation implies consistent labelling of bonds in the ring. The configurations of odd rings are labelled by the two classes of odd permutations of S_4, each containing six elements

$(\alpha\beta) \in$  and $(\alpha\beta\gamma\delta) \in$  ,

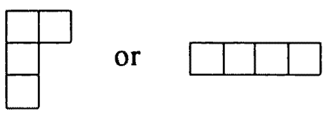
so that odd rings have two distinct lowest-energy configurations (one particular permutation in each class, selected by labelling (edge-colouring) the network and by choosing a spanning tree), separated by a large gauge transformation. On the torus representative of an odd loop, it is easy to label consistently (colour, identity permutation) all even rings and impossible to do so for odd rings. Change of labelling does not violate consistency. Since one odd ring is the sole, independent representative of a whole odd line, we have proven that the odd line has two distinct ground states, separated by a large gauge transformation.

Moreover, two different odd loops are independent since the bonds closing their representative odd rings are independent. The remaining rings are even, and can all be coloured consistently, i.e labelled by the identity permutation. Gauge transformation (a permutation) on one loop does not affect the class of permutation representing the structure of another (see eq. (2)). (The fiber bundle satisfies the "disjoint union" axiom in topological field theory. Labels for two disjoint boundaries of the manifold (odd lines) are independent.) However, the reader may check that the class of permutation is the same for all rings threaded through by the same odd line, provided that the even rings connecting them are in their ground state. (This is illustrated schematically in fig. 3.)

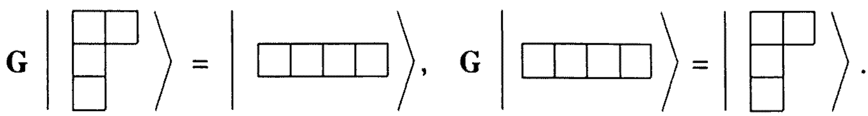
5.3. LARGE GAUGE TRANSFORMATION AND TUNNELLING

The algorithm chosen to compute the class of the covering transformation is physically arbitrary. It certainly depends on the choice and the labelling of the spanning tree. (A different spanning tree may have different "natural" labelling of the bonds.) Even with the same spanning tree, it is easy to imagine another choice labelling $(\alpha\beta)$ a ring formerly labelled $(\alpha\beta\gamma\delta)$. A change of algorithm, and of class, is therefore a large gauge transformation, leaving invariant the physical properties of the system, notably its energy (neglecting, as we have done, the real energy cost in bond-bending – we have only investigated the geometrical consequences of bond-bending energy, the reflection part of the rotatory reflection connection). (The spanning tree is only a means of labelling the network. A similar situation occurs with simplicial decomposition of manifolds [11,12]. Defects (loci of non-triviality of the fiber bundle in the base space) and physical properties are independent of the triangulation.)

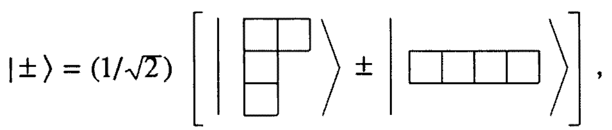
Thus, neither classical ground-state configuration



are gauge-invariant. Each is transformed into the other by a large gauge transformation **G**,



However, the physical configurations must be gauge-invariant,



(3)

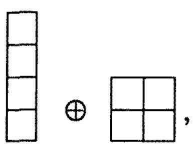
with one sign for the ground state and the other for the first excited state. Tunnelling, however slow, must take place to restore gauge invariance. This fully confirms fig. 1, which can also be obtained from different representations of glass, as a disordered elastic continuum [7,3], or as the result of a random sequence of decurving operations [11]. These quantum mechanical tunnelling modes between the two classical sectors of fig. 1, split by $h/2\pi$ times the tunnelling rate, have been observed in SQUIDS and in glasses. Their full identity has been revealed by a combination of four different types of experiments: specific heat, thermal conduction or sound propagation, saturation or (nonlinear) ultrasonic attenuation and echoes [1].

Note that the geometry of the base space (rings are either even or odd, odd rings form loops) does not match exactly the dynamics – geometry of the full bundle or homotopy (in which even rings have trivial ground states, whereas odd rings can be in either one of two states of twistedness or vorticity). Non-trivial geometry (frustration, odd line, in the base space) is only the source of non-trivial dynamics, which also depends on the nature of the dynamical variables (tetrapods) through the connection, i.e. on the full bundle.

For a random network with $z = 3$ which serves as a model for a-As, the odd permutations of S_3 belong to one class only: there can be no microscopic tunnelling modes in a-As, which is an assembly of sheets, loosely bound together, rather than a three-dimensional solid. (Moreover, rotation about a shared bond is an odd permutation for $z = 3$, so that the congruent transformation through the bond may have determinant 1. It always has determinant -1 for even z only.)

A $z = 4$ network (random froth) can also be the scaffolding for a model of disordered condensed matter in three dimensions at a meso- or macroscopic level, since vertex coordination of at least 4 is needed to span three-dimensional space, and no more than four cells are incident on one vertex if the structure is random. The above results which depend on $z = 4$ are therefore general for 3D glasses (as are tunnelling modes), even if the constituting atoms are not 4-valent.

Consider now *even* rings and the group A_z of even permutations. The identity forms a class (and a subgraph) by itself, and describes the ground state of the network restricted to even rings. It also guarantees consistent labelling of the bonds. It is clear that excitations are described by a subgroup of permutations. See, for example, the argument leading to eq. (2). The group structure describes how the state of one ring is affected by excitation of another. For permutation groups, a permutation and its inverse belong to the same class (since a class is uniquely given by cyclic decomposition of the permutation (Young tableau) and an inverse permutation has its cycles reversed). Invariant subgroups are therefore important to restrict the set of excitations. The identity is an invariant subgroup, restricting even rings in glass to their groud state. A_z itself includes all possible excitations. For $z \geq 5$, there is no further restriction (since all $A_{z\geq 5}$ are simple (Galois, Abel)). Only $z = 4$ has an invariant subgroup of order 4,



which may constitute an interesting subset of excitations in window glass or amorphous silicon.

We have seen an example of a discrete fiber bundle which is non-trivial because of *topological* disorder (frustration or odd lines). Discreteness is a physical attribute of the material, not only a useful mathematical artifice. Discrete fiber bundles (with discrete gauge groups as fiber) have been introduced only recently in field theory [4], but also as a guide [13] in decurving the ideal state of glass, polytope $\{3, 3, 5\}$ [11, 14]. (Since $\{3, 3, 5\}$ is a discrete scaffolding for the 3-sphere S^3, which is itself the total space of the (non-trivial: $S^3 \neq S^2 \times S^1$) Hopf bundle with gauge group S^1, one can illustrate concepts in the continuum in a small (120 points) discrete total space, and vice versa. For example, the fibers are geodesics (circles) which wind around each other with winding number 1, indicating the non-triviality of the bundle.)

6. Final remarks and conclusions

Glass can be represented as a non-trivial fiber bundle, which is the mathematical support of a gauge theory. The base space is the structural scaffolding, a random network, on which are put the physical quantities; it is the ordinary space. It contains the minimal, simplest conceivable line "defects", the odd loops, which are the geometric loci of non-triviality of the bundle. Disorder manifests itself in the base space at two levels: the distribution of odd lines is random and, it is argued, semi-dilute. Moreover, any generative symmetry has disappeared. The space group is trivial, and even the involution associated with the bonds (reflection) never adds up to a uniform action. It is indeed the action of this involution which differentiates odd and even rings.

It is in the fiber that the physical characteristics of the material are to be found, and one would have expected solid state physicists to take charge and flood the subject with words, acronyms and particular cases. Not at all. Large gauge transformation (topological curvature), the discrete shift along the fiber which is the physical manifestation of non-triviality of the bundle, is, like its source in base space, universal. It is an involution. It has spectacular (and direct: fig. 1) physical consequences in the low temperature behaviour of glass.

Odd loops, sources of non-triviality of the bundle and of tunnelling modes, have been observed indirectly by etch-pits [15] and, probably, directly in electron microscopy by Shang and co-workers [16].

From the near-perfect homogeneity of glass, emphasized in Lonsdale's quotation (section 2), we can infer the density of odd loops: they are "semi-diluted". This expression (borrowed from polymer solutions) means that there is only one coherence length in the assembly, the average distance between non-adjacent loop elements, which is the same whether they belong to the same or to different loops. (Both dilute (alphabet soup) and dense (bundles of loops) solutions show features with two distinct correlation lengths, which would give an observable signature in X-ray or neutron analysis, not seen up to now.) The unique coherence length in window glass is of the order of 4 nm; it separates short-range order and long-range homogeneity.

Large gauge transformations, spanning trees, and line defects are perhaps more familiar in continuous materials like superconductors. The phase of the complex superconducting order parameter plays the part of the current in an electrical network. It is defined everywhere in the superconductor, except at the core of vortices, and changes by 2π upon encircling a vortex (fixed, compatible configuration of the order parameter in the material). The phase is therefore *uniquely* defined everywhere, *arcwise* from a given origin, through the analogue in the continuum of the spanning tree of electrical network theory. A different selection of arcs around vortices (change of spanning tree) is a large gauge transformation, which modifies the phase at a point by a multiple of 2π.

References

[1] S. Hunklinger and A.K. Raychaudhari, *Progress in Low Temperature Physics*, Vol. IX, ed. D.F. Brewer (North-Holland, Amsterdam, 1986) p. 265;
W.A. Phillips (ed.), *Amorphous Solids. Low-Temperature Properties* (Springer, Berlin, 1981).

[2] G. Toulouse, Commun. Phys. 2(1977)115.
[3] N. Rivier, in: *Geometry in Condensed Matter Physics*, ed. J.F. Sadoc (World Scientific, Singapore, 1990) p. 1.
[4] N.S. Manton, Commun. Math. Phys. 113(1987)341.
[5] N. Rivier, Phil. Mag. A40(1979)859.
[6] H.J. Bernstein and A.V. Phillips, Sci. Am. 245(1991)94.
[7] D.M. Duffy and N. Rivier, J. Phys. Coll. (Paris) 43(1982)C9-475;
D.M. Duffy, Thesis, University of London, UK (1981);
N. Rivier, in: *Amorphous Materials: Modelling of Structure and Properties*, ed. V. Vitek (AIME, New York, 1983) p. 81.
[8] N. Rivier, Adv. Phys. 36(1987)95.
[9] P.N. Keating, Phys. Rev. 145(1966)637;
R. Alben, D. Weaire, J.E. Smith, Jr. and M.H. Brodsky, Phys. Rev. B11(1975)2271.
[10] G. Kirchhoff, Pogg. Ann. Phys. 72(1847)32;
N. Biggs, *Algebraic Graph Theory* (Cambridge University Press, Cambridge, 1974).
[11] J.F. Sadoc and N. Rivier, Phil. Mag. B55(1987)537.
[12] Regge calculus; see C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973) ch. 42.
[13] S. Nicolis, R. Mosseri and J.F. Sadoc, Europhys. Lett. 1(1986)571.
[14] M. Kléman and J.F. Sadoc, J. Phys. Lett. (Paris) 40(1979)569.
[15] A. Chenevas-Paule, in: *Semiconductors and Semimetals*, ed. J. Pankowe (Academic Press, New York, 1984) ch. 12.
[16] B.X. Liu, C.H. Shang and H.D. Li, J. Phys. Condens. Matter 3(1991)5769.